Some Elementary Mathematics of Integral Value Transformations & Its Associated Dynamical Systems

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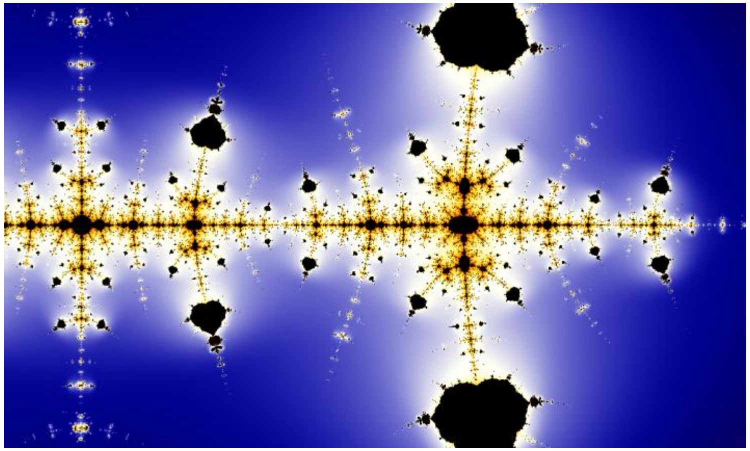

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Notations

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$\mathbb{N}_p = \{0, 1, 2, \ldots \mathrm{p} - 1\}$

p: Base of the number system.

k: # of variables.

j: Function number.

For $\mathrm{p} = 2$: Boolean Functions : An well known theory in Mathematics.

Denote: A $\mathrm{k} -$ variables Boolean Function as $\mathrm{f_j}^{2,\mathrm{k}}$.

Let, $f_j^{p,k}: \mathbb{N}_p^{\ k} \rightarrow \mathbb{N}_p$

We have defined an Association Function named as Integral Value Transformation $\Gamma: \mathfrak{S}^{p,k} \rightarrow \mathfrak{T}^{p,k}$

$\mathfrak{S}^{p,k}$ denote the set of all p-adic, k-variables functions.

$$\Gamma\left(f_j^{p,k}\right) = IVT_j^{p,k}$$

Therefore, $\mathfrak{T}^{p,k}$ is the set of all p-adic, k dimensional

Integral Value Transformations (IVTs).

Definition of IVT

$IVT^{p,k}{}_{j} : \mathbb{N}_0{}^{K} \rightarrow \mathbb{N}_0$

$IVT^{p,k}{}_{j}((n_1, n_2, \ldots . n_k)=$

$(f_j(a_0{}^{n_1}, a_0{}^{n_2}, \ldots, a_0{}^{n_k})\, f_j(a_1{}^{n_1}, a_1{}^{n_2}, \ldots, a_1{}^{n_k}) \ldots \ldots f_j(a_{l-1}{}^{n_1}, a_{l-1}{}^{n_2}, \ldots, a_{l-1}{}^{n_k}))_p = m$

where $n_1 = (a_0{}^{n_1} a_1{}^{n_1} \ldots . a_{l-1}{}^{n_1})_p$, $n_2 = (a_0{}^{n_2} a_1{}^{n_2} \ldots . a_{l-1}{}^{n_2})_p, \ldots \ldots n_k = (a_0{}^{n_k} a_1{}^{n_k} \ldots . a_{l-1}{}^{n_k})_p$

$f_j\colon \{0, 1, 2, \ldots, p-1\}^k \rightarrow \{0,1,2, \ldots, p-1\}.$

m is the decimal conversion from the p adic number.

In case of IVTs, The domain of action is extended from $\mathbb{N}_p$ to$\mathbb{N}_0$.

An Example

For p=3, k=1. There are 3^{3^1} number of 1-dimension 3-variable functions two of which are given in the table below :

Variables	f_7	f_{16}
0	0	1
1	2	2
2	1	1

$x = 55 = (2001)_3$

$IVT^{3,1}{}_7(x) = (f_7(2)\, f_7(0) f_7(0) f_7(1))_3 = (1002)_3 = 29$

$IVT^{3,1}{}_{16}(x) = (f_{16}(2)\, f_{16}(0) f_{16}(0) f_{16}(1))_3 = (1112)_3 = 41$

Set of One Dimensional IVTs

Let us fix the domain of IVTs as N_0 (k=1) and thus the above definition boils down to the following:

$$IVT^{p,1}_j(x) = \left(f_j(x_n)\, f_j(x_{n-1}) \dots \dots \dots f_j(x_1)\right)_p = m$$

where m is the decimal conversion from the p adic number, and $x = (x_n\, x_{n-1} \dots \dots x_1)_p$.

Let us denote the set of $IVT^{p,1}_j$ as

$$T^{p,1} = \left\{ IVT^{p,1}_j \; : N \to N \;\middle|\; \begin{array}{c} 0 \le j < p^p,\ IVT^{p,1}_j(x) = \left(f_j(x_n)\, f_j(x_{n-1}) \dots \dots \dots f_j(x_1)\right)_p = m \\ \text{where m is the decimal conversion from the p adic number} \\ \text{and } x = (x_n\, x_{n-1} \dots \dots x_1)_p \end{array} \right\}$$

Algebraic Structures on $T^{p,1}$

$(T^{p,1}, \oplus, \otimes)$ forms a **Commutative Ring** under the operations $\oplus$ and $\otimes$ defined as

$(IVT^{p,1}{}_{j_1} \oplus IVT^{p,1}{}_{j_2})(x) = \left(f_{j_1}(x_n) \oplus_p f_{j_2}(x_n) \quad f_{j_1}(x_{n-1}) \oplus_p f_{j_2}(x_{n-1}) \dots \dots \dots f_{j_1}(x_1) \oplus_p f_{j_2}(x_1)\right)_p$ and

$(IVT^{p,1}{}_{j_1} \otimes IVT^{p,1}{}_{j_2})(x) = \left(f_{j_1}(x_n) \otimes_p f_{j_2}(x_n) \quad f_{j_1}(x_{n-1}) \otimes_p f_{j_2}(x_{n-1}) \dots \dots \dots f_{j_1}(x_1) \otimes_p f_{j_2}(x_1)\right)_p$

where $x = (x_n \; x_{n-1} \dots \dots x_1)_p$ and $\oplus_p$ denotes addition modulo p and $\otimes_p$ denotes multiplication modulo p.

But the **multiplicative inverse of non-zero IVTs does not exist.** So We have failed to proceed to **Field Structure**.

An Attempt to Make Field Structure

We already know that $(T^{p,1}, \oplus)$ where $\oplus$ is defined as

$$(\,IVT^{p,1}{}_{j_1} \oplus IVT^{p,1}{}_{j_2})(x) = \Big(f_{j_1}(x_n) \oplus_p f_{j_2}(x_n)\;\; f_{j_1}(x_{n-1}) \oplus_p f_{j_2}(x_{n-1}) \ldots\ldots\ldots f_{j_1}(x_1) \oplus_p f_{j_2}(x_1)\Big)_p$$

where $x = (x_n\; x_{n-1} \ldots\ldots x_1\,)_p$ is an abelian group. Our task is now to define an operation $\otimes$ in such a way that $(T^{p,1}, \otimes)$ forms an abelian group and follows the distributive laws.

Let us define an operation $\otimes$ as below:

$\otimes$: $T^{p,1}$ X $T^{p,1} \rightarrow T^{p,1}$ defined as

$$(\,IVT^{p,1}{}_{j_1} \otimes IVT^{p,1}{}_{j_2})(x) = \Big(f_{j_1}(x_n) \otimes_p f_{j_2}(x_n)\;\; f_{j_1}(x_{n-1}) \otimes_p f_{j_2}(x_{n-1}) \ldots\ldots\ldots f_{j_1}(x_1) \otimes_p f_{j_2}(x_1)\Big)_p$$

if either $IVT^{p,1}{}_{j_1}$ = Identity or $IVT^{p,1}{}_{j_2}$ = Identity or both

else $(\,IVT^{p,1}{}_{j_1} \otimes IVT^{p,1}{}_{j_2})(x) =$

$$\Big(\delta\big(f_{j_1}(x_n) \otimes_p f_{j_2}(x_n)\big)\;\; \delta(f_{j_1}(x_{n-1}) \otimes_p f_{j_2}(x_{n-1})) \ldots\ldots\ldots \delta(f_{j_1}(x_1) \otimes_p f_{j_2}(x_1))\Big)_p$$

where $\delta(x_i) = \begin{cases} x_i, & \text{if } x_i \neq 0 \\ 1, & \text{if } x_i = 0 \end{cases}$ where $x_i \in \mathbb{F}_p$

$(T^{p,1}, \otimes)$ is an abelian group but it does not follow the distributive properties and hence fails to be a field. Thus, unfortunately our efforts in this direction have been unsuccessful so far.

Immediate Expectations

- Would it be at all possible to have Field Structure on the set of IVTs?
- **Main Problem:** Distributive Law (Compatibility of Binary Operations).

So, Is there any protocol to design to make two binary operations Compatible (Distributive) on a set of interest?

- **Basic Question:** How can we ensure the **Existence or Non-Existence** about the structure of Group, Ring, Field on a set of interest?

Vector Space and Module Structure

$(T^{p,1}{}_{\#}\ ,\oplus\ ,\wedge\ )$ forms a **Vector space over a field** $\mathbb{F}_p$(**p is prime**) where ^ denotes scalar multiplication defined as $(c\wedge IVT^{p,1}{}_j)(x)=\left(c \otimes_p f_j(x_n) \quad c \otimes_p f_j(x_{n-1}) \ldots\ldots\ldots \quad c \otimes_p f_j(x_1)\right)_p$

where $x = (x_n\ x_{n-1} \ldots\ldots x_1\ )_p$ and $\otimes_p$ denotes multiplication modulo p.

Remark: When p is prime, $T^{p,1}$is a finite vector space (with p^pfunctions) with $\dim(T^{p,1})$ = p over the finite field $\mathbb{F}_p$. When p is composite number, $(T^{p,1},\oplus\ ,\wedge\ )$ forms a module over $\mathbb{F}_p$ which is a commutative ring with unity under addition and multiplication modulo p. Moreover, it is a *free module* since it has a basis which will be shown in the next slide.

Basis Functions for The Vector Space

The basis of $(T^{p,1}, \oplus, \wedge)$ is $\{ IVT^{p,1}_j$ such that $j=p^i$ where $i=0,1,2,\ldots p-1 \}$.

Thus for any $IVT^{p,1}_j \in T^{p,1}$, $IVT^{p,1}_j = a_0 \wedge IVT^{p,1}_{p^0} + a_1 \wedge IVT^{p,1}_{p^1} + a_2 \wedge IVT^{p,1}_{p^2} \ldots\ldots\ldots\ldots + a_{p-1} \wedge IVT^{p,1}_{p^{p-1}}$

where $a_i \in \mathbb{F}_p$ and $j=\sum_{i=0}^{p-1} a_i\, p^i$.

Illustration:

Basis functions of $T^{3,1}$ are $IVT^{3,1}_1$, $IVT^{3,1}_3$, $IVT^{3,1}_9$

In $T^{3,1}$, we can write $IVT^{3,1}_{21} = a_0 \wedge IVT^{3,1}_{3^0} + a_1 \wedge IVT^{3,1}_{3^1} + a_2 \wedge IVT^{3,1}_{3^2}$ where $a_i \in \mathbb{F}_3$, $i=0,1,2$

and $21=a_0 + 3a_1 + 9a_2$. So we must have $a_0 = 0$, $a_1 = 1$, $a_2 = 2$.

Thus we must have

$IVT^{3,1}_{21} = 0 \wedge IVT^{3,1}_{3^0} + 1 \wedge IVT^{3,1}_3 + 2 \wedge IVT^{3,1}_9$

Now,

- We have a way expressing any function in the p-adic function space in terms of the basis functions of the p-adic space.

- Next we would like to see the relation between functions in the p-adic and (p+1)-adic function spaces. To do this , we have devised a mechanism by defining a transformation between the bases sets of each space which can be extended to the whole space.

p-adic to (p+1)-adic basis functions

Let us define a function $T : B_p \cup \{ IVT^{p,1}{}_0 \} \rightarrow B_{p+1}$ as

$$T(IVT^{p,1}{}_j) = \begin{cases} IVT^{(p+1),1}{}_{(p+1)^p} & \text{if} \quad j = 0 \\ IVT^{(p+1),1}{}_{(p+1)^i} & \text{if } j = p^i \mid i = 0,1,\ldots p-1 \end{cases}$$

B_p and B_{p+1} are the bases of $T^{p,1}$ and $T^{p+1,1}$ respectively and

$f^{(p+1),1}{}_j : \{0,1,2,\ldots\ldots p-1, p\} \rightarrow \{0,1,2,\ldots\ldots\ldots p-1, p\}$ is defined as

$$f^{(p+1),1}{}_j(p) = \begin{cases} 1 & \text{if} \quad f^{(p+1),1}{}_j(i) = 0 \ \forall\, i = 0,1,2,\ldots p-1 \\ 0 & \text{if} \quad f^{(p+1),1}{}_j(i) = 1 \text{ for some } i = 0,1,2,\ldots p-1. \end{cases}$$

Then T is an Isomorphism.

An Illustration for the basis functions

Through the Transformation , $T : B_2 \cup \{ IVT^{p,1}{}_0 \} \rightarrow B_3$

, we have

$$IVT^{2,1}{}_0 \leftrightarrow IVT^{3,1}{}_9$$

$$IVT^{2,1}{}_1 \leftrightarrow IVT^{3,1}{}_1$$

$$IVT^{2,1}{}_2 \leftrightarrow IVT^{3,1}{}_3$$

Extension of T leads to...

Through an extension of the transformation T, we get $IVT^{2,1}{}_3 \rightarrow IVT^{3,1}{}_4$.

We first express $IVT^{2,1}{}_3$ in terms of the basis functions and then apply the transformation T on it,

$$IVT^{2,1}{}_3 = IVT^{2,1}{}_1 + IVT^{2,1}{}_2$$

$$T(IVT^{2,1}{}_3) = T(IVT^{2,1}{}_1 + IVT^{2,1}{}_2)$$

$$= IVT^{3,1}{}_1 + IVT^{3,1}{}_3 = IVT^{3,1}{}_4$$

Some properties of IVT

There are **p** number of linear functions in $T^{p,1}$ since for any linear function $f_\#$, we must have $f_\#$ (0) =0 and $f_\#(1) = i$ for $i \in \{0,1,2,\ldots\ldots p-1\}$ for a p-adic system. Consequently we would have $f_\#(r) = r \otimes_p f_\#(1)$ where $0 \leq \# \leq (p\text{-}1)$.

There are **p**! number of bijective functions in $T^{p,1}$.

All the p^p functions are surjective but not all of them would be injective. For a p-adic function , there are p number of variables i.e, 0,1,2,……..p-1 and an injective function say f_j would have to take one of these variables again to one of these variables and hence we would have p! number of possibilities of arrangements of these variables.

Thus, so far some of the algebraic properties of the space of Integral Value Transformations have been highlighted .

Now, we will explore some analytical notions like derivability of these functions.

Derivability of IVT's

- For any function, to introduce the notion of derivability at a point, we first talk of the neighborhood of a point .

- Since IVT's are discrete functions, the neighborhood around a point will be a discrete set. Let us see the difference in the concept of neighborhood in the continuous and discrete cases.

Neighborhood around a point x_0

In case neighbourhood of a point in a continuous domain, it's defined as below

$N(x_0, r) = \{x_0 \in X : d(x, x_0) < r\}$ where d denotes the metric defined on the set X.

Now, for $x_0 \in \mathbb{N}_0$, the neighbourhood of x_0 consisting of discrete points boils down to the following

$N(x_0, r) = \{x \in \mathbb{N}_0 : |x - x_0| < r\} =$

$\{x_0 - r + 1, \dots\dots\dots . x_0 + r - 1\}; r > 1, r \in \mathbb{N}_0$

Next we define an operator on the space of functions for derivability at a point.

Differentiation

To define an operator, D for differentiation, the following fundamental rules should be satisfied.

1) Linearity: $D(IVT^{p,1}{}_{j_1} + IVT^{p,1}{}_{j_2}) = D(IVT^{p,1}{}_{j_1}) + D(IVT^{p,1}{}_{j_2})$

2) Homogeneity: $D(k\text{^}IVT^{p,1}{}_{j_1}) = k\, D(IVT^{p,1}{}_{j_1})$

3) Leibnitz Rule: $D(IVT^{p,1}{}_{j_1} . IVT^{p,1}{}_{j_2}) = IVT^{p,1}{}_{j_1} . D(IVT^{p,1}{}_{j_2}) + IVT^{p,1}{}_{j_2} . D(IVT^{p,1}{}_{j_1})$

Now we are in a position to define the operator D on $(T^{p,1}{}_{\#}, \oplus, \wedge)$

We first define a left derivative for an $IVT^{p,1}{}_j$ at an arbitrary point $c \in \mathbb{N}_0$ as the following:

Let $LD(IVT^{p,1}{}_j)(c)$ denote the *left derivative* of the function $IVT^{p,1}{}_j$ at the point c defined as

$$LD(IVT^{p,1}{}_j)(c) = \min_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{IVT^{p,1}{}_j(x) - IVT^{p,1}{}_j(c)}{(x-c)} \right\}$$

Let $RD(IVT^{p,1}{}_j)(c)$ denote the *right derivative* of the function $IVT^{p,1}{}_j$ at the point c defined as

$$RD(IVT^{p,1}{}_j)(c) = \max_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{IVT^{p,1}{}_j(x) - IVT^{p,1}{}_j(c)}{(x-c)} \right\}.$$

If both the right and left derivative of the function $IVT^{p,1}{}_j$ exist at the point c and are equal, then we say that $IVT^{p,1}{}_j$ is differentiable at the point c and the derivative at c is equal to $D(IVT^{p,1}{}_j)(c)$.

We see that….

Linearity:

$$LD(\ IVT^{p,1}{}_{j_1} + IVT^{p,1}{}_{j_2})(c) = LD(\ IVT^{p,1}{}_{j_1})(c) + LD(IVT^{p,1}{}_{j_2})(c)$$

Leibnitz Rule:

$$LD(\ IVT^{p,1}{}_{j_1}.\ IVT^{p,1}{}_{j_2})(c)$$

$$= \min_{x:x\epsilon N(c,\varepsilon)}\{\ IVT^{p,1}{}_{j_1}(x)\ \}\ LD(\ IVT^{p,1}{}_{j_2})(c) + IVT^{p,1}{}_{j_2}(c).\ LD(\ IVT^{p,1}{}_{j_1})(c)$$

Similarly, the above holds for the right hand derivative and are equal if the function is derivable.

However, Homogeneity does not hold from the fact that

$$LD(\ k\wedge IVT^{p,1}{}_{j})(c) = \min_{x:x\epsilon N(c,\varepsilon)}\left\{\frac{k\wedge IVT^{p,1}{}_{j}(x) - k\wedge IVT^{p,1}{}_{j}(c)}{(x-c)}\right\}$$

$$= \min_{x:x\epsilon N(c,\varepsilon)}\left\{\frac{k\wedge IVT^{p,1}{}_{j}(x) - k\wedge IVT^{p,1}{}_{j}(c)}{(x-c)}\right\}$$

$$= LD(IVT^{p,1}{}_{j_1})(c) \text{ for some } j_1.$$

An illustration

Consider, $IVT^{2,1}{}_2(x) = x$

Let c be any arbitrary natural number,

$$LD(IVT^{2,1}{}_2)(c) = \min_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{IVT^{2,1}{}_2(x) - IVT^{2,1}{}_2(c)}{(x-c)} \right\}$$

$$= \min_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{x-c}{(x-c)} \right\} = 1$$

$$RD(IVT^{2,1}{}_2)(c) = \max_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{IVT^{2,1}{}_2(x) - IVT^{2,1}{}_2(c)}{(x-c)} \right\}$$

$$= \max_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{x-c}{(x-c)} \right\} = 1$$

Therefore $IVT^{2,1}{}_2$ is *everywhere differentiable* as both $LD(IVT^{2,1}{}_2)(c)$ and $RD(IVT^{2,1}{}_2)(c)$ exist and are equal and the derivative is 1.

Another example..

$IVT^{2,1}{}_1(x) = (2^s - 1) - x$ where x has a s-bit representation

$$LD(IVT^{2,1}{}_1)(c) = \min_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{IVT^{2,1}{}_1(x) - IVT^{2,1}{}_1(c)}{(x-c)} \right\}$$

$$= \min_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{(2^{s_1}-1) - x - [(2^{s_2}-1) - c]}{(x-c)} \right\} = \min_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{(2^{s_1} - 2^{s_2})}{(x-c)} - 1 \right\} \rightarrow -1$$

$$RD(IVT^{2,1}{}_1)(c) = \max_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{IVT^{2,1}{}_1(x) - IVT^{2,1}{}_1(c)}{(x-c)} \right\}$$

$$= \max_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{(2^{s_1}-1) - x - [(2^{s_2}-1) - c]}{(x-c)} \right\} = \max_{x:x\epsilon N(c,\varepsilon)} \left\{ \frac{(2^{s_1} - 2^{s_2})}{(x-c)} - 1 \right\} \rightarrow +\infty$$

As $LD(IVT^{2,1}{}_1)(c)$ and $RD(IVT^{2,1}{}_1)(c)$ are not equal therefore $IVT^{2,1}{}_1$ is *nowhere differentiable*.

What is now?

As of now we have done some rudimentary algebraic and analytical structures on the set of IVTs, which enable us to study some analytical work namely **Dynamical Systems of IVTs**.

Discrete Dynamical System

Let G be a semi-group with respect to f and U be any space. Let

T : G X U $\rightarrow U$ defined as

T(g,x) = T_g(x) such that $f\left(T_g, T_h\right) = T_{f\,(g,h)}$

where $T_g : U \rightarrow U$

Then (G, T) is called a dynamical system.

If G = $\mathbb{N}_0$ or Z, then it's called a discrete dynamical system.

Discrete Dynamical System of IVT

Let us define

$T : \mathbb{N}_0 \; X \; \mathbb{N}_0 \; \rightarrow \mathbb{N}_0$ as

$T(n, x) \; = \; T_n(x)$

Where $T_n(x) = (IVT^{p,1}{}_j)^n(x) = \underbrace{[(IVT^{p,1}{}_j)\,(IVT^{p,1}{}_j) \ldots \ldots .\,(IVT^{p,1}{}_j)]}_{n\ times}\;(x)$

$(\mathbb{N}_0 , T)$ is a discrete dynamical system.

$T(n, x)$ is the evolution function and n is the evolution parameter of the dynamical system.

X_0 is the initial state and $\mathbb{N}_0$ is the state/phase space. Corresponding to each p and j, we get a different dynamical system.

Collatz Conjecture (1937)

Who is Collatz?

Lothar Collatz was a German mathematician. In 1937 he posed the famous **Collatz conjecture**, **during his PhD**.

Prof. Collatz *was convinced that mathematics and mathematicians had a responsibility to apply their results to, and be motivated by, real world phenomena. He never wearied of fighting for this conviction.*

(6th July 1910-26th Sept.1990)

Collatz Conjecture

Consider a function from $\mathbb{N}_0$ to $\mathbb{N}_0$, defined as follows:

$$\mathrm{T}(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (3n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

Next define the iterate of T as usual:

$$\begin{cases} \mathrm{T}^{(0)}(n) = n \\ \mathrm{T}^{(i+1)}(n) = \mathrm{T}(\mathrm{T}^{i}(n)) \end{cases}$$

The question is now to show that for every $n \in \mathbb{N}_0$, there is a finite k, such that

$$\mathrm{T}^{(k)}(n) = 1.$$

This is the conjecture also known as the 3n + 1 conjecture

An illustration

A straightforward example: take $n = 7$, then we have the following sequence

$$
\begin{array}{cccccccccccccccccccccccc}
7 & \to & 11 & \to & 17 & \to & 26 & \to & 13 & \to & 20 & \to & 10 & \to & 5 & \to & 8 & \to & 4 & \to & 2 & \to & 1 \\
0 & & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 & & 9 & & 10 & & 11
\end{array}
$$

therefore $\mathrm{T}^{(11)}(7) = 1$ and $k = 11$.

Now we are free to play with some other natural numbers as we wish.

It is worth to be aware of the facts:

- The conjecture remains *Unanswered,* although it has been shown to be valid for all natural numbers up to 5.764 $*10^{18}$.
- Professor Paul ErdŐs once commented on the Collatz conjecture: "Mathematics is not yet ready for such problems”.

Generalizations and Analogue Conjectures

Conjecture in the domain of Rational Numbers

Let θ be the following function on $\mathbb{Q}_+$, the set of nonnegative rational numbers:

$$\theta(x) = \begin{cases} (x-1)/3 & x \geq 1 \\ 2x/(1-x) & x < 1 \end{cases}$$

We make the following conjecture:

Conjecture *For every $x \in \mathbb{Q}_+$, there exists $n \geq 1$ so that $\theta^n(x) = 0$.*

Further generalization

$$g(n) = \begin{cases} \frac{2}{3}n, & \text{if } n \equiv 0 \pmod 3, \\ \frac{4}{3}n - \frac{1}{3}, & \text{if } n \equiv 1 \pmod 3, \\ \frac{4}{3}n + \frac{1}{3}, & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

Conjecture: Consider an iterative scheme $g^k(n) = g\left(g^{k-1}(n)\right) : k > 0$

There exists a positive integer k such that $g^k(n) = 1.$

This is how we can make the general conjecture in modulo n system.

A Collatz type Conjecture of Prof. Mridul K. Sen

$$g(n) = \frac{\varphi(n)}{2} : \text{n is even and} > 2$$

$$g(n) = 1 : n = 2$$

$$g(n) = \frac{3n+1}{2} : \text{n is odd}$$

Conjecture: Consider an iterative scheme $g^k(n) = g\left(g^{k-1}(n)\right) : k > 0$

There exists a positive integer k such that $g^k(n) = 1.$

So where is the challenge?

Challenge: To define an analogue Conjecture and

to prove or disprove it!

This, in our view, will be one step forward towards the settlement of Collatz Conjecture in the sense that we may get some clue for settling the original conjecture.

Let us accept the Challenge.

Let us consider the iterative scheme $(IVT^{p,1}{}_j)^i(n) = IVT^{p,1}{}_j\left((IVT^{p,1}{}_j)^{i-1}(n)\right)$

Conjecture: there exists an integer i such that $(IVT^{p,1}{}_j)^i(n) = 0$

Let us see the above conjecture in case of $p = 2, j = 1$

Iterative convergence for few numbers

X_0	*Iterative sequences*
0	0
1	0
2	1, 0
3	0
4	3, 0
5	2, 1, 0
6	1, 0
7	0
8	7, 0
9	6, 1, 0

10	5, 2, 1, 0
11	4, 3, 0
12	3, 0
13	2, 1, 0
14	1, 0
15	0
16	15, 0
17	14, 1, 0
18	13, 2, 1, 0
19	12, 3, 0
20	11, 4, 3, 0

Main Result of the Collatz like Conjecture in IVT

The iterative scheme $\{X_n\}$ converges to 0 for any given X_0 where $X_{n+1} = IVT^{2,1}{}_1(X_n)$

Lemma:

(I) *For any non-negative integer of the form* $X_0 = 2^n + P$, $IVT^{2,1}{}_1(X_0) = 2^n - (P+1)$ *for some non-negative integer.*

(II) *For any non-negative integer of Merseene form* $X_0 = 2^n - 1$, $IVT^{2,1}{}_1(X_0) = 0$.

Main Proof is based on Mathematical Induction

The iterative scheme $\{X_n\}$ converges to 0 for any given X_0 where $X_{n+1} = IVT^{2,1}{}_1(X_n)$

Proof:
We use Strong Mathematical Induction (SMI) principle to prove the theorem. Let us consider a set T_n, a set of X_0s which is defined as $\{2^n + p: 0 \leq p \leq 2^n - 1\}$. Clearly for $n = 0, T_0 = \{1\}$, for $n = 1, T_1 = \{2, 3\}, for\ n = 2, T_2 = \{4, 5, 6, 7\}$ and so on…

In this way, all natural numbers along with zero could be captured by the said scheme. Let us define ${T_1}^n$ as a set of $IVT^{2,1}{}_1(X_0)$s corresponding to n for T.

For $n = 0, T_0 = \{1\}$ and so readily ${T_1}^0 = \{0\}$ i.e. X_0 converges to 0 by Lemma-2.2.1 (II).

For $n = 1$, by lemma-2.2.1 (I) and (II) $T_1 = \{2, 3\}$ becomes ${T_1}^1 = \{1, 0\} = T_0 \cup {T_1}^0$. Already, $T_0\ and\ {T_1}^0$ have converged to 0.

For $n = 2$, by the lemma 2.2.1 (I) and (II) in $T_2 = \{4, 5, 6, 7\}$ becomes ${T_1}^2 = \{3, 2, 1, 0\} = T_1 \cup {T_1}^1$. Previously, $T_1\ and\ {T_1}^1$ have converged to 0.

Let us hypothesize that the theorem be true for all $n = m$.

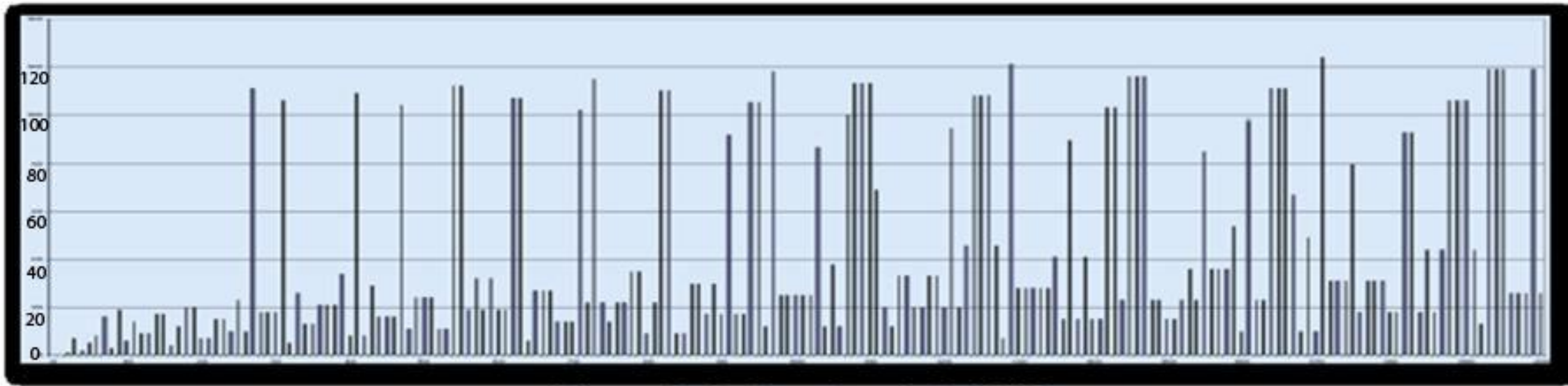
Let us try to establish the theorem is true for $n = m + 1$ also.

Now, ${T_1}^{m+1} = \{2^{m+1} + p: 0 \leq p \leq 2^{m+1} - 1\}$ then ${T_1}^{m+1} = \{p: 0 \leq p \leq 2^{m+1} - 1\} = T_m \cup {T_1}^m$. According to the SMI hypothesis we could say the iterative scheme is converging to 0.

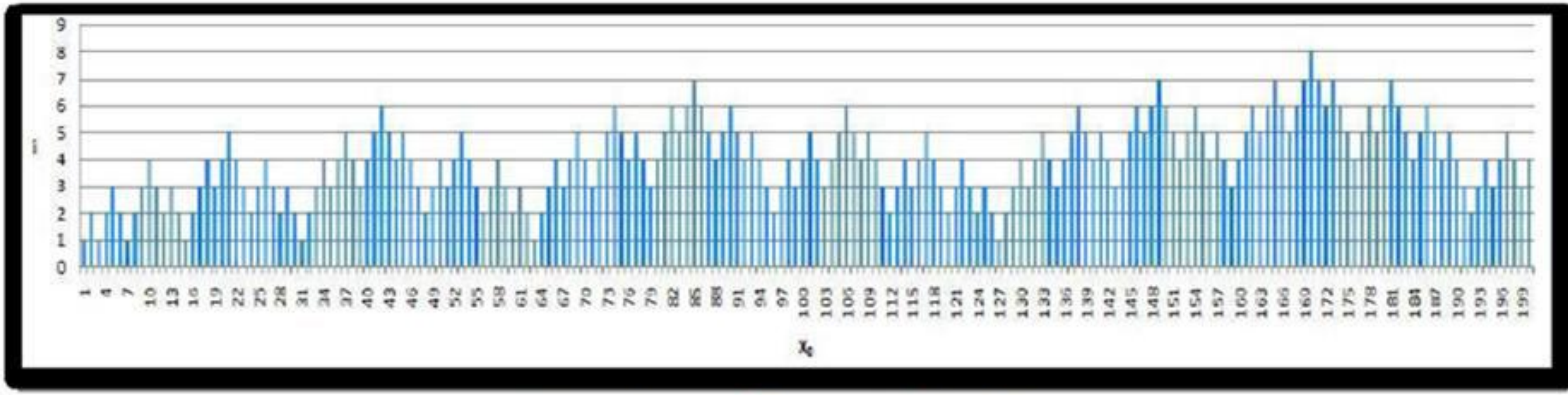
Therefore the required theorem is followed. (*Proved*).

In the subsequent section we would explore the convergence behavior of the iterative scheme corresponding to $IVT^{2,1}{}_1$ and Collatz function.

Convergence behavior of Collatz and IVT



[Figure 1: Collatz Graph (1-200)]



[Figure 2: $IVT^{2,1}{}_1$ Graph (1-200)]

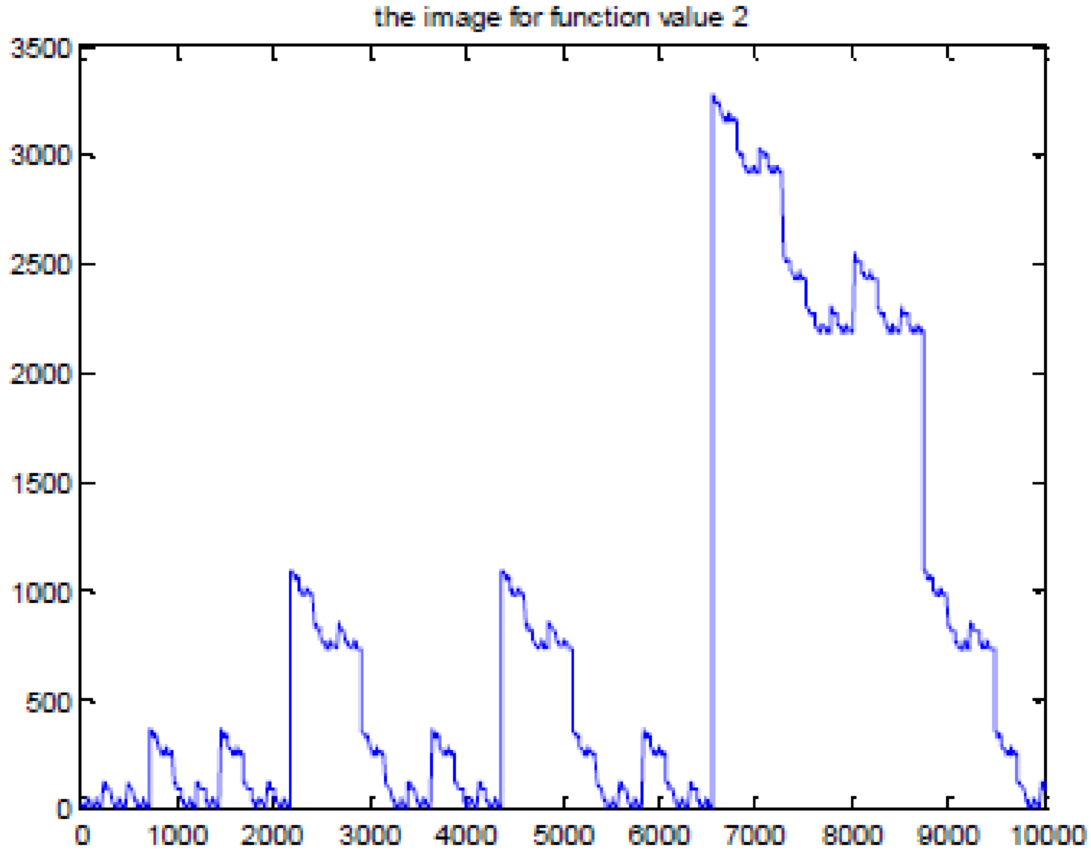
Affine Discrete Dynamical Systems

Consider

- $X_{n+1} = A.\,IVT_{\#}{}^{p,1}(X_n), n \geq 0 \ldots (P_1)$
- $X_{n+1} = A.\,IVT_{\#}{}^{p,1}(X_n) + B, n \geq 0 \ldots (P_2)$
- $X_{n+1} = IVT_{\#}{}^{p,1}(a.\,X_n), n \geq 0 \ldots (P_3)$
- $X_{n+1} = IVT_{\#}{}^{p,1}(a.\,X_n + b), n \geq 0 \ldots (P_4)$

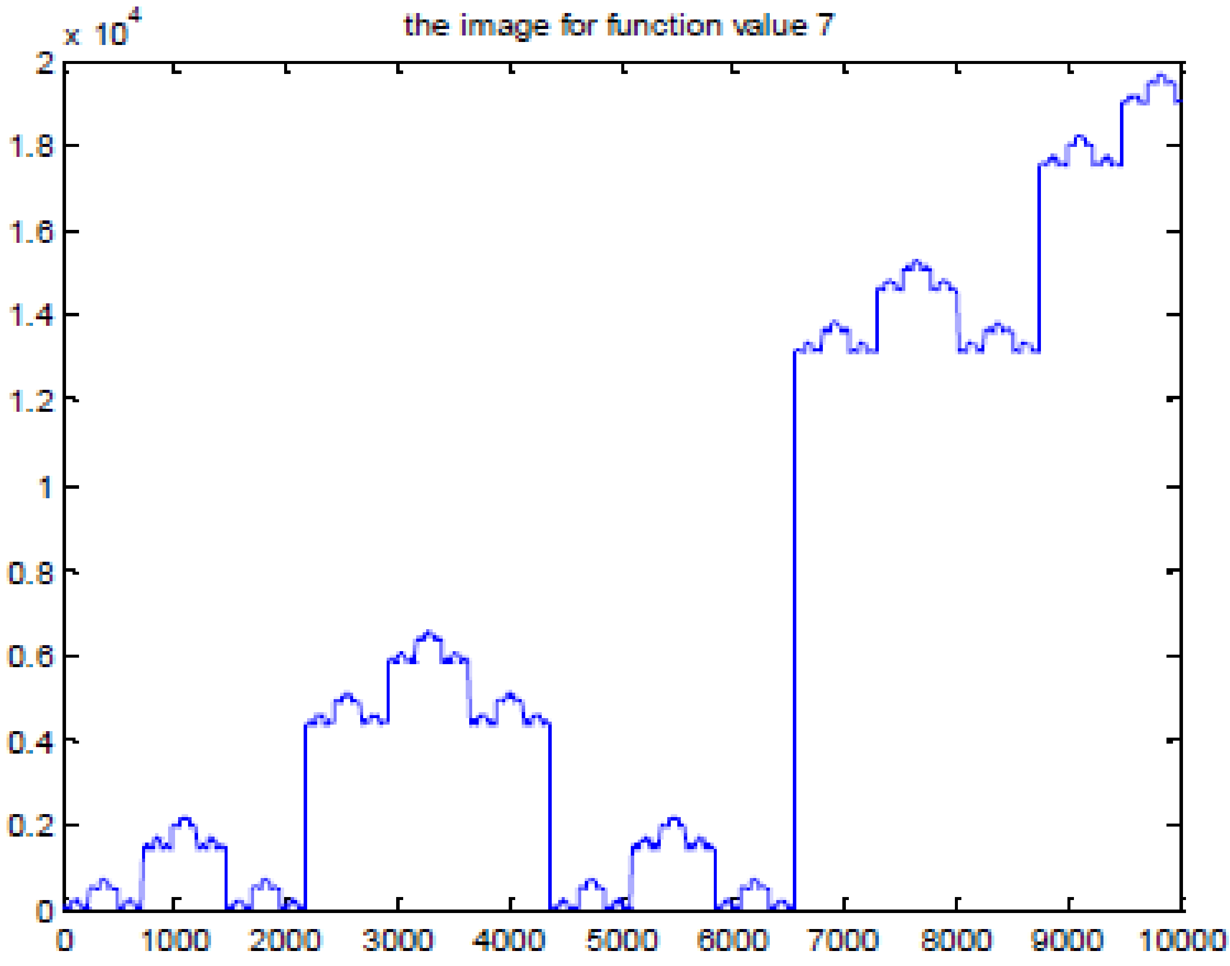
$$a, A, b, and\ B\ are\ fixed\ Natural\ numbers.$$

Graph of IVTs as 1st Iteration



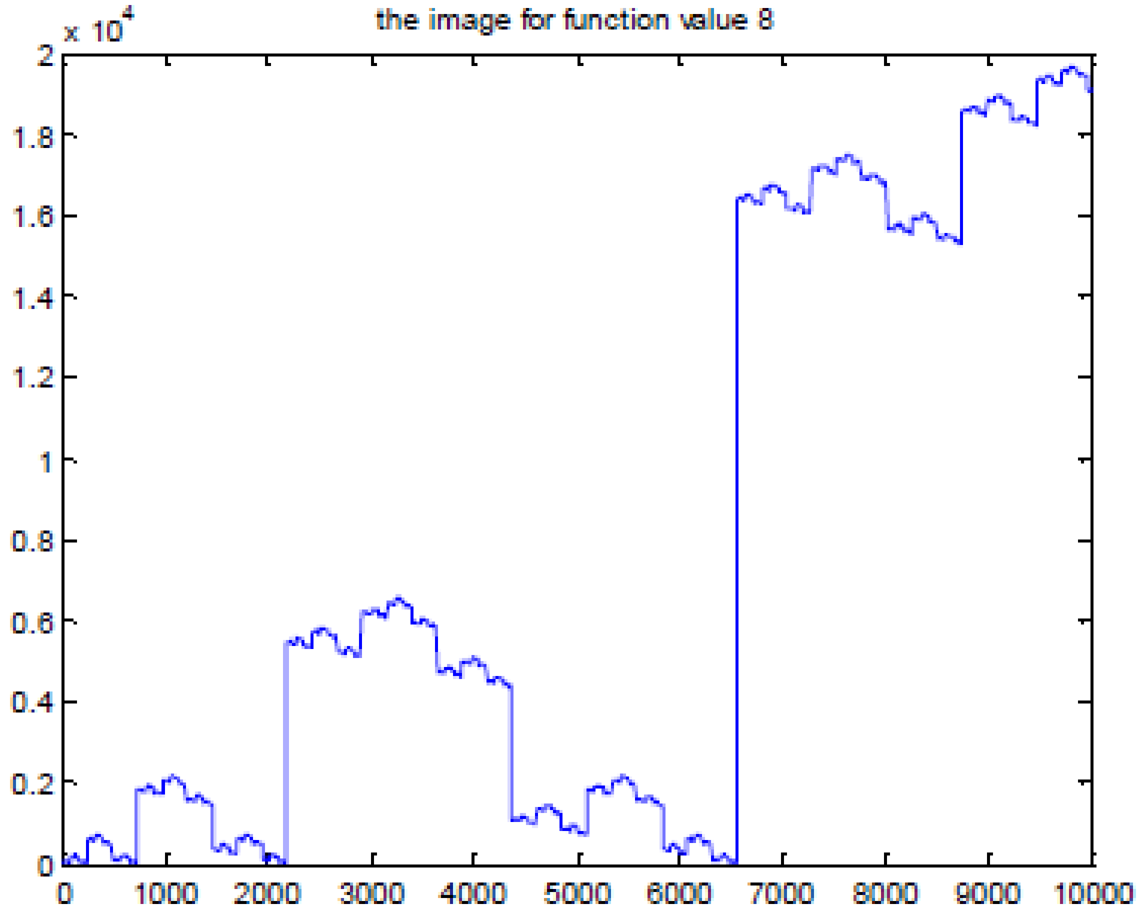
$$Y_1 \sim Y_0 \sim Y_0 \mathrm{IVT}_1^{3,1}$$

Graph of IVTs as 1st Iteration



$Y_1 \sim Y_0$ of $\mathrm{IVT}_6^{3,1}$

Graph of IVTs as 1st Iteration



$Y_1 \sim Y_0$ of $\mathrm{IVT}_7^{3,1}$

Inferences…

The graphs are nowhere differentiable and self-repeating (self-similar), imply that Collatz-like ADDS forms *fractal*. The fractal dimensions of these four $IVT_j^{p,1}$s (j = 0, 1, 6, 7) are 1, 1.94006, 1.94012 and 1.94016 respectively.

Similar graphs can be obtained for the system $y_1 = IVT_j^{p,1}(A y_0 + B)$ and those graphs will have same pattern.

Non-Periodicity, Non-Linearity of ADDS

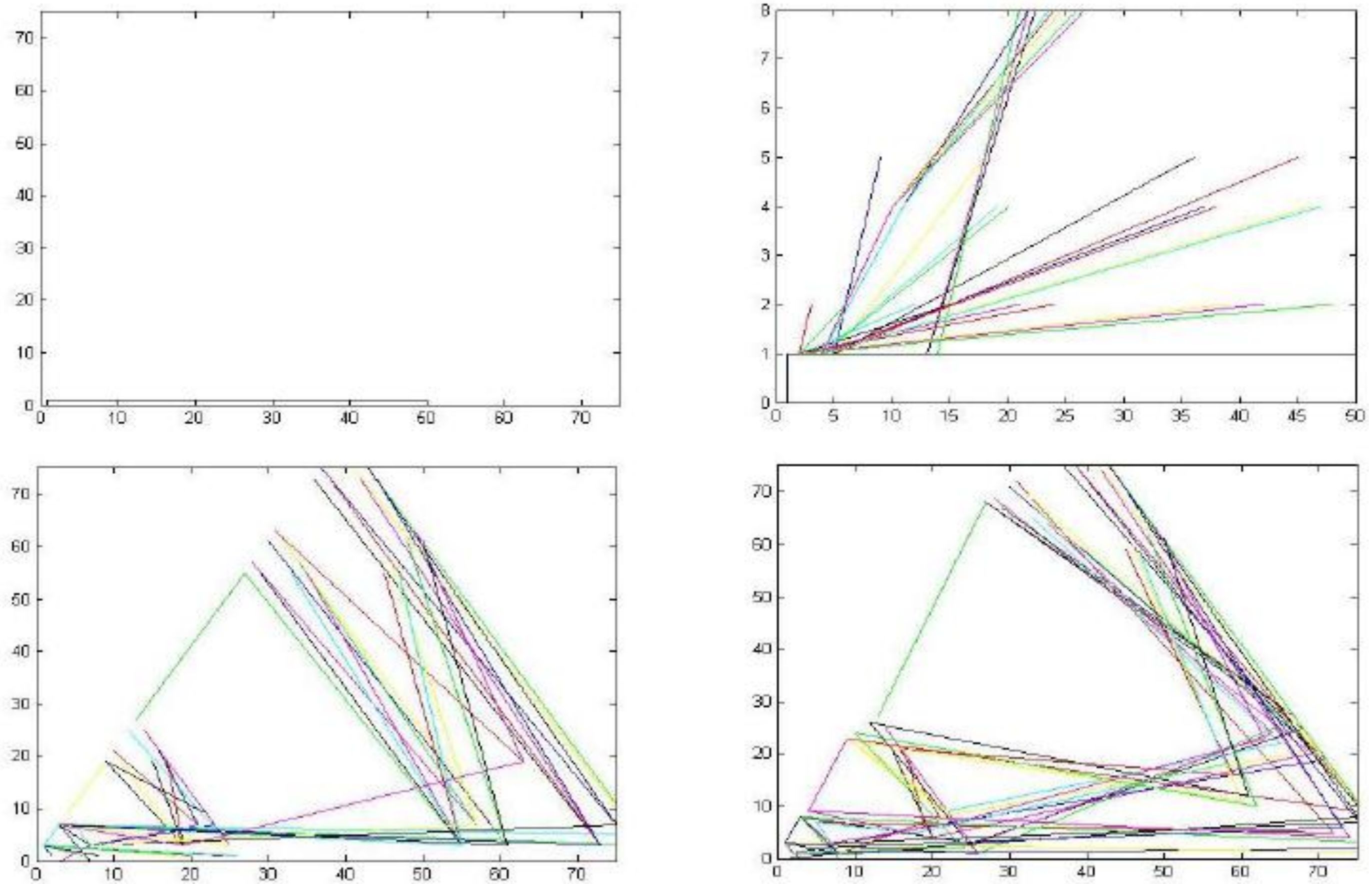


Fig. 1: *Graph of* $Y_{i+1} \sim Y_i$ *of* $IVT_0^{3,1}$, $IVT_1^{3,1}$, $IVT_6^{3,1}$ *and* $IVT_7^{3,1}$

Current Problems and Scopes

Suppose $\mathrm{IVT_{\#}}^{\mathrm{p,1}}\colon\ \mathbb{N}_0 \rightarrow \mathbb{N}_0$

$$\text{Consider, } \mathrm{X_{n+1}} = \mathrm{IVT_{\#}}^{\mathrm{p,1}}(\mathrm{X_n}), \mathrm{n} \geq 0$$

Given that for every $\mathrm{X_0} \in \mathbb{N}_0$, there exists a $\mathrm{m} \in \mathbb{N}$ such that $\mathrm{X_m} = l$.
Let us call the above hypothesis as $\mathrm{P_0}$

$\boldsymbol{Case}$: $\mathbf{P_1}$

$$X_{n+1} = A.\,IVT_{\#}{}^{p,1}(X_n), n \geq 0$$

Let $IVT_{\#}{}^{p,1}(aX) = a\ IVT_{\#}{}^{p,1}(X)$

Therefore, $X_{n+1} = A^{n+1}\ (IVT_{\#}{}^{p,1})^{n+1}(X_0)$

Since, P_0 is given to be true,

So there exists a natural number m such that $(IVT_{\#}{}^{p,1})^{n+1}(X_0) = l$

$$\text{Therefore, } X_m = A^m\ (IVT_{\#}{}^{p,1})^m(X_0) = A^m l.$$

Similar Cases for P_2, P_3, & P_4

Let $\mathrm{IVT_{\#}}^{\mathrm{p,1}}(\mathrm{aX}+\mathrm{bY}) = a\,\mathrm{IVT_{\#}}^{\mathrm{p,1}}(X) + b\,\mathrm{IVT_{\#}}^{\mathrm{p,1}}(Y)$

Therefore, $\mathrm{X_n} = A^n\,(\mathrm{IVT_{\#}}^{\mathrm{p,1}})^n(\mathrm{X_0}) + \left[\frac{\{A.\mathrm{IVT_{\#}}^{\mathrm{p,1}}(1)\}^n - 1}{A.\mathrm{IVT_{\#}}^{\mathrm{p,1}}(1) - 1}\right] B + B$

Since, P_0 is given to be true,

So there exists a natural number m such that $(\mathrm{IVT_{\#}}^{\mathrm{p,1}})^{n+1}(\mathrm{X_0}) = l$

Therefore, $\mathrm{X_m} = A^m l + \left[\frac{\{A.\mathrm{IVT_{\#}}^{\mathrm{p,1}}(1)\}^m - 1}{A.\mathrm{IVT_{\#}}^{\mathrm{p,1}}(1) - 1}\right] B + B$ for $\mathbf{P_2}$

$$\mathrm{X_m} = a^m\,(\mathrm{IVT_{\#}}^{\mathrm{p,1}})^m(\mathrm{X_0}) = a^m l \text{ for } \mathbf{P_3}$$

$$\mathrm{X_m} = a^m l + \left[\frac{\{a.\,\mathrm{IVT_{\#}}^{\mathrm{p,1}}(1)\}^m - 1}{a.\,\mathrm{IVT_{\#}}^{\mathrm{p,1}}(1) - 1}\right] b.\,\mathrm{IVT_{\#}}^{\mathrm{p,1}}(1) \text{ for } \mathbf{P_4}$$

Observations

- Linearity of IVTs
- $The\ X_m$ iteration for every intial value

X_0 in the cases for P_1, P_2, P_3, P_4 can be obtained whenever P_0 is given to be true.

Question & Our Conviction…

* Does there exists at all a single attractor for every initial values corresponding to such of the ADDS?

The X_m iteration for every initial value X_0 **depends** on m (the number of iterations) for each case.

Therefore, it is our strong belief that …

There is a **Countable Basin of Attractors** for each of the ADDS of IVTs.

Some Unsolved Problems Yet!

- In Case of Non-linear function, How should we care the cases?
- The cases, we have raised so far are up to **Affine Level**. We have option to extend up to **any polynomial of degree** of our interest.

In those cases, how do we handle once the case P_0 is known.

Time to Wrap Up…

This Mathematics of IVTs is one-fold research work of Home of Mathematical Genomics (HMG-ISI Kolkata) (www.isical.ac.in/~hmg).

For references one may visit publication page at www.isical.ac.in/~hmg

Thanks

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